

Period function and characterizations of Isochronous potentials

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Abstract

We are interested at first in the study of the monotonicity for the period function of the conservative equation (1) $\ddot{x} + g(x) = 0$. Some refinements of known criteria are brought. Moreover, we give necessary and sufficient conditions so that the analytic potential of equation (1) is isochronous. These conditions which are different from those introduced firstly by Koukles and Piskounov and thereafter by Urabe appear sometime to be easier to use. We then apply these results to produce families of isochronous potentials depending on many parameters, some of them are news. Moreover, analytic isochronicity requirements of parametrized potentials will also be considered

Key Words and phrases: period function, monotonicity, isochronicity, center, polynomial systems.¹.

1 Introduction and statement of results

Consider the scalar equation with a center at the origin 0

$$\ddot{x} + g(x) = 0 \tag{1}$$

or its planar equivalent system

$$\dot{x} = y, \quad \dot{y} = -g(x) \tag{2}$$

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where $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$ and $g(x)$ is analytic on R .
Let $G(x)$ be the potential of equation (1)

$$G(x) = \int_0^x g(\xi) d\xi.$$

We suppose in the sequel the following hypothesis holds

$$(\mathcal{H}) \quad \begin{cases} \text{There exist } a < 0 < b \text{ such that } G(a) = G(b) = c, \ G(x) < c \\ \text{and } xg(x) > 0 \text{ for all } a < x < b \text{ and } x \neq 0. \end{cases}$$

Moreover, without lose of generality we will assume in the sequel that $g(0) = 0$ and $g'(0) = 1$.

Under these conditions system (2) admits a periodic orbit in the phase plane with energy c and a, b are the turning points of this orbit satisfying inequalities $\bar{a} < a < 0$, $0 < b < \bar{b}$, for some real \bar{a} and \bar{b} .

This means the origin 0 is a center of (2). This center is isochronous when the period of all orbits near $0 \in R^2$ are constant ($T = \frac{2\pi}{\sqrt{g'(0)}} = 2\pi$). The corresponding potential $G(x)$ is also called isochronous.

Let $T(c)$ denotes the minimal period of this periodic orbit depending on the energy. It can be expressed under the following form

$$T(c) = \sqrt{2} \int_a^b \frac{dx}{\sqrt{c - G(x)}}. \quad (3)$$

The period function is well defined for any c such that $0 < c < \bar{c}$ and when $\bar{c} < +\infty$ then \bar{c} is the energy of the homoclinic orbit with $\bar{a} < 0 < \bar{b}$ its turning points. It is well known that $T(c)$ is an analytic function of c .

Since the potential $G(x)$ has a local minimum at 0 , then we may consider an involution A by

$$G(A(x)) = G(x) \text{ and } A(x)x < 0$$

for all $x \in [a, b]$. So, any closed orbit is A -invariant and A exchanges the turning points: $b = A(a)$.

In fact, $A(x)$ is well defined in the interval $[a, b]$. To see that, set the function

$$\rho(x) = \frac{x - A(x)}{2}.$$

This function is such that $\rho(A(x)) = -\rho(x)$ and $\rho'(x) = \frac{1-A'(x)}{2}$. Since $A'(x) < 0$ we get $\rho'(x) > 0$ and therefore ρ is an analytic diffeomorphism on $[a, b]$. Then $A(x) = \rho^{-1}(-\rho(x))$ is well defined.

Conversely, by using this involution we may calculate for a prescribed period function $T(c)$ the distance between the turning points. Indeed, Following Landau and Lifchitz we have [9, Chap.3, 12.1]):

$$T(c) = \sqrt{2} \int_0^c \left[\frac{dA(a)}{dG} - \frac{da}{dG} \right] \frac{dG}{\sqrt{c-G}}.$$

which implies

$$A(a) - a = b - a = \frac{1}{\pi\sqrt{2}} \int_0^c \frac{T(\gamma)d\gamma}{\sqrt{c-\gamma}}. \quad (4)$$

When the potential (or the center 0) is isochronous or equivalently the period is constant for all orbits near zero one has necessarily

$$b - a = 2\sqrt{2c}.$$

The behavior of the period function is important in applied mathematics. Many authors ([3],[4],[10],[11]) are interested in the monotonicity of this period function as well as in isochronicity cases of equation (1) ([6],[7],[10]). Isochronous centers plays a central role in several problems of dynamical systems. Also, it intervenes in the investigation of the quantum spectrum. Many studies have been devoted to the problem of the relationship between isochronicity equivalence and quantum isospectrality condition. See [1],[6] for more details.

In many papers one tried to characterize isochronous potentials. In the case of the rational potentials, it can be shown [2] that the only rational potential with a constant period which not reduces to a polynomial is the isotonic one

$$G(x) = \frac{1}{8\alpha^2} \left[\alpha x + 1 - \frac{1}{\alpha x + 1} \right]^2.$$

The paper is organized as follows :

- In the first part we study the monotonicity of the period function of a closed orbit of equation (1) depending on the energy and we produce better criteria than those known. In particular, we complete the work started by Chow and Wang [5].

- We propose to analyze the notion of isochronicity in using a new track. We will provide necessary and sufficient condition so that the center 0 of system (2) be isochronous, or equivalently its analytic potential is isochronous. Many applications are brought.

- In the analytic case, let us write the potential under the following form

$$G(x) = \frac{1}{2}x^2 + G_1(x) + G_2(x)$$

where $G_1(x)$ is odd and $G_2(x)$ is even. It is wellknown that the only symmetric isochronous potential is the harmonic one (i.e. $G(x) = \frac{1}{2}x^2$). We prove here that when $G(x)$ is isochronous G_1 and G_2 are related and consequently $G_2(x) \equiv 0$ implies necessarily $G_1(x) \equiv 0$.

Moreover, a simple proof of Chalykh-Veselov result [2] concerning rational potentials will be provided.

- Finally, we derive a three-parameters family of isochronous potential more general than the one given by Dorignac [6]. This family includes harmonic and isotonic and Urabe potentials as well as the Bolotin-Mac Kay potential. Other families of isochronous potentials also will be considered.

2 The period function

Let us consider the period function of equation (1) depending on the energy $T \equiv T(c)$. When g is an analytic function defined for $x \in [a, b]$, $T(c)$ is also an analytic function defined for $c \in]0, \bar{c}]$ such that $\lim_{c \rightarrow 0^+} T(c) = 2\pi$. The following result states sufficient conditions for $T(c)$ to be monotonic.

Theorem A *Let $g(x)$ be an analytic function and $G(x) = \int_0^x g(\xi)d\xi$ be the potential of equation (1) $\ddot{x} + g(x) = 0$. Suppose hypothesis (\mathcal{H}) holds, let us define the n -polynomial with respect to G ,*

$$f_n(G) = f(0) + f'(0)G + \frac{1}{2}f''(0)G^2 + \dots + \frac{1}{n!}f^{(n)}(0)G^n$$

where $f(0) = -g''(0)/3 = -(1/3)\frac{d^2g}{dx^2}(0)$, $f'(0) = -(7/9)g'''(0) + g^{(4)}(0)/5$, $f''(0) = -\frac{1}{21}g^{(6)}(0) - \frac{310}{54}g^{''5}(0) + 2g^{''2}(0)g^{(4)}(0), \dots$

Suppose that for a fixed $n \in N$ and for $x \in [0, b]$ one has

$$(\mathcal{C}_n) \quad \frac{d}{dx} \left[\frac{G}{g^2}(x) \right] > f_n(G) > \frac{d}{dx} \left[\frac{G}{g^2}(A(x)) \right]$$

$$(or \quad \frac{d}{dx} \left[\frac{G}{g^2}(x) \right] < f_n(G) < \frac{d}{dx} \left[\frac{G}{g^2}(A(x)) \right])$$

then the period function of (1) $T'(c) > 0$ (or < 0) for $0 < c < \bar{c}$.

Proof of Theorem A Recall that

$$T(c) = \sqrt{2} \int_a^b \frac{dx}{\sqrt{c - G(x)}}.$$

In order to study the monotonicity of the period function $T(c)$ depending on the energy, it is sometime convenient to study its derivatives. We need the following

Lemma 2-1 *The derivative of the period function (depending on the energy) $T'(c) = \frac{dT}{dc}$ may be written*

$$T'(c) = \left(\frac{1}{c\sqrt{2}} \right) \int_a^b \frac{g^2(x) - 2G(x)g'(x)}{g^2(x)\sqrt{c - G(x)}} dx \quad (5)$$

This lemma has been initially proved by Chow and Wang [5] but the proof we give below is different.

Proof Consider the change $G(x) = s^2$, where $s = u(x)$ is a function of x (this change has a sense because $G(x)$ is positive in a neighborhood of 0 according to the hypothesis $xg(x) > 0$). It yields

$$T(c) = 2\sqrt{2} \int_{-\sqrt{c}}^{\sqrt{c}} \frac{sds}{g(u^{-1}(s))\sqrt{c - s^2}}.$$

By another change $s = \sqrt{c} \sin \theta$ the period function may be expressed

$$T(c) = \sqrt{2} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{u'(u^{-1}(\sqrt{c} \sin \theta))} = \sqrt{2} \int_{-\pi/2}^{\pi/2} (u^{-1})'(\sqrt{c} \sin \theta) d\theta.$$

Then we get by another way the derivative of the period function

$$T'(c) = \frac{2}{\sqrt{c}} \int_{-\pi/2}^{\pi/2} (u^{-1})''(\sqrt{c} \sin \theta) \sin \theta d\theta.$$

By splitting $T'(c) = \frac{2}{\sqrt{c}} \int_{-\pi/2}^0 + \frac{2}{\sqrt{c}} \int_0^{\pi/2}$ and using the formula $(u^{-1})''(u(s)) = -\frac{u''(s)}{(u'(s))^3}$ and making another time the change of variable $s = u(x) = \sqrt{c} \sin \theta$ we then obtain

$$\begin{aligned} T'(c) &= \frac{1}{c\sqrt{2}} \int_a^0 \frac{g^2(x) - 2G(x)g'(x)}{g^3(x)} \frac{-g(x)}{\sqrt{c - G(x)}} dx + \\ &\quad \frac{1}{c\sqrt{2}} \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{g^3(x)} \frac{g(x)}{\sqrt{c - G(x)}} dx. \end{aligned}$$

Since $xg(x) < 0$ the derivative can be written

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{g^2(x) - 2G(x)g'(x)}{g^2(x)\sqrt{c - G(x)}} dx.$$

To prove Theorem A observe at first

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c - G(x)}} g(x) dx. \quad (6)$$

Write

$$(c\sqrt{2})T'(c) = \int_a^0 \frac{g^2(x) - 2G(x)g'(x)}{g^2(x)\sqrt{c - G(x)}} dx + \int_0^b \frac{g^2(x) - 2G(x)g'(x)}{g^2(x)\sqrt{c - G(x)}} dx.$$

By using the involution A and since $x(g(x) > 0$ for $x \neq 0$ we get

$$\begin{aligned} T'(c) &= \left(\frac{1}{c\sqrt{2}}\right) \int_b^0 \frac{\frac{d}{dx}(G/g^2)(A(x))}{\sqrt{c - G(x)}} g(A(x)) A'(x) dx + \left(\frac{1}{c\sqrt{2}}\right) \int_0^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c - G(x)}} g(x) dx \\ &= \left(\frac{1}{c\sqrt{2}}\right) \int_b^0 \frac{\frac{d}{dx}(G/g^2)(A(x))}{\sqrt{c - G(x)}} g(x) dx + \left(\frac{1}{c\sqrt{2}}\right) \int_0^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c - G(x)}} g(x) dx \end{aligned}$$

$$= \left(\frac{1}{c\sqrt{2}}\right) \int_0^b \frac{\frac{d}{dx}\left(\frac{G}{g^2}(x)\right) - \frac{d}{dx}\left(\frac{G}{g^2}(A(x))\right)}{\sqrt{c-G(x)}} g(x) dx.$$

Then

$$T'(c) > (or <) \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{f(0) + f'(0)G + \frac{1}{2}f''(0)G^2 + \dots + \frac{1}{n!}f^n(0)G^n}{\sqrt{c-G(x)}} g(x) dx.$$

Let us integrate by parts the following for $p \geq 1$

$$\int \frac{G^p}{\sqrt{c-G(x)}} g(x) dx = \int \frac{G^p}{\sqrt{c-G(x)}} dG = -G^p \sqrt{c-G} + 2p \int G^{p-1} \sqrt{c-G} dG.$$

Therefore, it is easy to see that the following integral should be zero since $G(a) = G(b) = c$

$$\int_a^b \frac{G^p}{\sqrt{c-G(x)}} g(x) dx = 0$$

for any integer $0 \leq p \leq n$.

Thus, the right side of (6) is non zero and condition

$$\frac{d}{dx}[G(x)/g^2(x)] - f_n(G) \neq 0$$

implies $T'(c) \neq 0$. This means $T = T(c)$ is monotonic.

As applications of Theorem A, we derive at first monotonicity conditions for the period function $T(c)$ depending on the energy. This problem holds importance in dynamical systems (see [4], [10] and [11] for example). In this part, we find again some known criteria for the monotonicity of the period function and propose some others which are better and seem to be new in the literature.

For a complete study and a comparison between these sufficients conditions we refer to [3] and [4] and references therein.

Notice that the monotonicity criteria for the period function produced by Cor. 2.5 of [5] is more general than those given by F. Rothe [10] and R. Schaaf [11].

We propose the following which slightly improves Cor. 2.5 of [5].

Corollary 2-2 Suppose hypothesis (\mathcal{H}) holds and let $g(x)$ be an analytic function for $\bar{a} < x < \bar{b}$ and $G(x) = \int_0^x g(\xi)d\xi$ be the potential of (1) and $g''(0) \neq 0$.
1 - Suppose condition (\mathcal{C}_0) holds, this means

$$g^2(x) + \frac{g''(0)}{3}g^3(x) - 2G(x)g'(x) > 0(\text{or } < 0)$$

then the period function of (1) is such that $T'(c) > 0(\text{or } < 0)$ for $0 < c < \bar{c}$.
2 - Suppose

$$\frac{d}{dx}[G(x)/g^2(x)] = \alpha \quad (7)$$

where $\alpha = -g''(0)/3 \neq 0$ then 0 is an isochronous center of (1).
Moreover, the so-called Urabe potential

$$G(x) = \frac{4}{\alpha^2} - \frac{2}{\alpha}\left(x + 2\frac{\sqrt{1-\alpha x}}{\alpha}\right)$$

where $|x| < 1/\alpha$ is the unique analytic solution of (7) verifying $g(0) = G(0) = 0$ and $G''(0) = 1$.

Proof of Corollary 2-1 Indeed, (7) may be written

$$g^2(x) + \frac{g''(0)}{3}g^3(x) - 2G(x)g'(x) = g^3(x)\frac{d}{dx}([G(x)/g^2(x)] + \frac{g''(0)}{3}g^3(x).$$

Then (7) must be

$$\frac{d}{dx}[G(x)/g^2(x)] = -g''(0)/3 = \alpha$$

Impliedy

$$G(x) = \frac{1}{2}g^2(x) + \alpha x g^2(x)$$

since $g'(0) = 1$. Thus, starting from (6) the derivative of the period function $T(c)$ may be expressed under the following forms

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{g^2(x) - 2G(x)g'(x)}{g^2(x)\sqrt{c - G(x)}} dx$$

$$\begin{aligned}
&= \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c-G(x)}} g(x) dx \\
&= \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\frac{d}{dx}(\sqrt{G}/g)^2}{\sqrt{c-G(x)}} g(x) dx.
\end{aligned}$$

Integrating by parts, one gets

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\alpha}{\sqrt{c-G(x)}} g(x) dx \equiv 0.$$

On the other hand, since

$$\frac{g^2 - 2Gg'}{g^3} = \frac{d}{dx}(G/g^2) = \frac{d}{dx}(\sqrt{G}/g)^2$$

it implies that $G = G(x)$ may be inverted and that x can be expressed

$$x = \sqrt{2G} + \alpha G. \quad (7')$$

Remarks 2-3 Inversing now equation (7') and according to initial conditions $g(0) = G(0) = 0$ and $G'''(0) = 1$ it is easy to see that the potential

$$G(x) = \frac{4}{\alpha^2} - \frac{2}{\alpha} \left(x + 2 \frac{\sqrt{1-\alpha x}}{\alpha}\right)$$

is the unique solution of (7') or (7). $G(x)$ is called *Urabe potential*.

Notice that its derivative $g(x)$ must also be a solution of

$$5(g''(x))^2 - 3g'(x)g'''(x) = 0$$

and consequently for any Urabe potential we get necessarily

$$g^{(4)}(0) = \frac{35}{9} g'''(0). \quad (8)$$

When in addition $g''(0) = 0$ (i.e. $\alpha = 0$) equation (7) has as unique solution the harmonic potential $G(x) = (1/2)x^2$.

In fact, as we will see below there exist isochronous potentials that do not verify the restrictive condition (8). Consider the following expression

$$\mathcal{F}(x) = g^2(x) + \frac{g''(0)}{3} g^3(x) - 2G(x)g'(x) + \left(\frac{7g'''(0)}{9} - \frac{g^{(4)}(0)}{5}\right) g^3(x)G(x). \quad (9)$$

The following brings a new monotonicity condition for the period function

Corollary 2-4 *Let $g(x)$ be an analytic function and $G(x) = \int_0^x g(\xi)d\xi$ be the potential of equation (1). Suppose (\mathcal{C}_1) holds, this means*

$$g^2(x) + \frac{g''(0)}{3}g^3(x) - 2G(x)g'(x) + \left(\frac{7g'''(0)}{9} - \frac{g^{(4)}(0)}{5}\right)g^3(x)G(x) > 0 \text{ (or } < 0)$$

for $\bar{a} < x < \bar{b}$ then $T'(c) > 0$ (or < 0) for $0 < c < \bar{c}$. So a sufficient condition for (1) to have an isochronous center is

$$\mathcal{F}(x) = g^2(x) + \frac{g''(0)}{3}g^3(x) - 2G(x)g'(x) + \left(\frac{7g'''(0)}{9} - \frac{g^{(4)}(0)}{5}\right)g^3(x)G(x) = 0$$

or equivalently

$$\frac{d}{dx}[G(x)/g^2(x)] = \alpha + \beta G$$

where $\alpha = -\frac{g''(0)}{3}$ and $\beta = -\left(\frac{7g'''(0)}{9} - \frac{g^{(4)}(0)}{5}\right) \neq 0$.

Proof of Corollary 2-4 Indeed, recall that

$$\frac{g^2 - 2Gg'}{g^3} = \frac{d}{dx}(G/g^2) = \frac{d}{dx}(\sqrt{G}/g)^2$$

Then $\mathcal{F}(x) \equiv 0$ means that

$$\frac{g^2 - 2Gg'}{g^3} = \alpha + \beta G$$

where $\alpha = -g''(0)/3$ and $\beta = -(7/9)g'''(0) + g^{(4)}(0)/5$. Consider again the derivative of the period function

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c - G(x)}} g(x) dx = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\frac{d}{dx}(\sqrt{G}/g)^2}{\sqrt{c - G(x)}} g(x) dx.$$

So one obtains

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\alpha + \beta G}{\sqrt{c - G(x)}} g(x) dx$$

$$= \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\alpha}{\sqrt{c-G(x)}} g(x) dx + \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\beta G}{\sqrt{c-G(x)}} g(x) dx.$$

By Corollary 2-2, the first integral of the right side should be 0. The second integral may be written as

$$\begin{aligned} & \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\beta G}{\sqrt{c-G(x)}} g(x) dx = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\beta(G-c)}{\sqrt{c-G(x)}} g(x) dx \\ & = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \beta \sqrt{c-G(x)} g(x) dx = \left(\frac{1}{c\sqrt{2}}\right) \int_{G(a)}^{G(b)} \beta \sqrt{c-G} dG = 0 \end{aligned}$$

Thus, one gets

$$T'(c) \equiv 0.$$

Moreover, we also may deduce

$$2 \frac{d}{dx}(\sqrt{G}/g) = \beta g \sqrt{G} + \alpha g / \sqrt{G}.$$

On the other hands $G = G(x)$ may be inverted and x can be expressed in terms of G . By integration and according to hypotheses $G(0) = g(0) = 0$ and $g'(0) = 1$, it yields

$$2(\sqrt{G}/g) = (2\beta/3)G\sqrt{G} + 2\alpha\sqrt{G} + \sqrt{2},$$

or equivalently

$$1 = (\beta/3)gG + \alpha g + \left(\frac{g}{\sqrt{2G}}\right)$$

So by another integration one obtains

$$\begin{aligned} x &= (\beta/6)G^2 + \alpha G + \sqrt{2G} \\ x &= \sqrt{2G} - \frac{g''(0)}{3}G + \left(\frac{7g'''(0)}{54} - \frac{g^{(4)}(0)}{30}\right)G^2. \end{aligned}$$

Remarks 2-5 Notice that clearly Corollary 2-4 is more general than Corollary 2-2. We then obtain a better criteria of the monotonicity for the period function. In the sense that (\mathcal{C}_0) implies (\mathcal{C}_1) which implies $T'(c) > 0$ (or < 0).

By the same way for any fixed n many other sufficient conditions ensuring the monotonicity of the period function of the form

$$g^2(x) - 2G(x)g'(x) - \alpha g^3(x) - \beta g^3(x)G(x) - \gamma g^3(x)G^2(x) - \dots > 0 \text{ (or } < 0)$$

may be deduced. In particular, condition

$$g^2(x) - 2G(x)g'(x) - \alpha g^3(x) - \beta g^3(x)G(x) - \gamma g^3(x)G^2(x) - \dots = 0$$

implies that the potential $G(x)$ is isochronous and x can be expressed

$$x = \sqrt{2G} + \alpha G + (\beta/6)G^2 + (\gamma/24)G^3 + \dots$$

Thus, for a n-polynomial $f_n(G) = f(0) + f'(0)G + \frac{1}{2}f''(0)G^2 + \dots + \frac{1}{n!}f^n(0)G^n$ conditions of Theorem A

$$(\mathcal{C}_n) \quad \frac{d}{dx}\left[\frac{G}{g^2}(x)\right] > (<) f_n(G) > (<) \frac{d}{dx}\left[\frac{G}{g^2}(A(x))\right]$$

ensuring $T'(c) > 0$ (or < 0) are such that (\mathcal{C}_{n+1}) is finest than (\mathcal{C}_n) .

3 Isochronicity conditions for a center of equation (1)

3.1 Isochronicity conditions for a center of equation (1)

The problem to determine whether the center is isochronous has attracted many researchers for long time. This problem has been recently revived due to advancement of computer algebra. New powerfull algorithms have been discovered indeed.

For the sake of completeness let us recall below different criteria for the isochronicity of periodical solutions of equation (1).

Using formula (4), Landau and Lifschitz [9] deduced the following

Proposition 3-1 [9, Chap.3] When $g(x)$ is continuous and hypothesis (\mathcal{H}) holds, (1) has an isochronous center at the origin 0 if and only if

$$x - A(x) = 2\sqrt{2G(x)}$$

for all $0 < x < b$ where A is the involution such that $G(A(x)) = G(x)$ and $A(x)x < 0$ for $x \neq 0$.

When $g(x)$ is a continuous function, $g(x)$ and x having the same sign, Koukles and Piskounov [7] produced necessary and sufficient conditions so

that the center of the system (2) is isochronous.

Proposition 3-2 [7, Th 5] *A set of necessary and sufficient conditions for the period of every solution of (1) near 0 to be equal to a constant T_0 is:*

- 1 - $g(x)$ is continuous and positive for small positive x .
- 2 - $\liminf_{x \rightarrow 0} \left| \frac{g(x)}{x} \right| \neq 0$.
- 3 - $T_0 \geq \limsup_{x \rightarrow 0} \frac{2\pi}{\sqrt{g(x)/x}}$.
- 4 - $g(-x) = -\frac{d}{dx} \left[\frac{T_0}{\pi} \sqrt{2x} - \left(\int_0^x g(u) du \right)^{-1} \right]^{-1}$ where index -1 denotes an inverse function.

This result was improved by Koukles and Piskounov themselves in the analytic case. They proved the following

Proposition 3-3 [7, Th 6] *Let $g(x)$ be a real analytic function. Then the center 0 of the equation (1) $\ddot{x} + g(x) = 0$ is isochronous if and only if the inverse function $x = \Theta(z)$*

$$\left[\int_0^x g(\xi) d\xi \right]^{-1} = \Theta(z)$$

is of the form

$$\Theta(z) = \sqrt{z} + P(z)$$

where P is a real analytic function such that $P(0) = 0$.

Later, Urabe proposed some refinements of Proposition 3-3 by considering the assumption of the differentiability of $g(x)$ at 0 and proved the following result which is most used than Propositions 3-2 and 3-3

Proposition 3-4 [13] *Let $g(x)$ be a C^1 function defined in V_0 a neighborhood of 0 verifying $xg(x) > 0$ in $V_0/\{0\}$. Then the system (2) has an isochronous center at the origin 0 if and only if $g(x)$ may be written*

$$g(x) = \frac{X}{1 + h(X)}$$

where $h(X)$ is a C^1 odd function and $X = \sqrt{2G(x)}$, $\frac{X}{x} > 0$ for $x \neq 0$.

We will propose an alternative approach in order to derive isochronous potentials. Some other criteria or equivalent characterizations will be presented. Their significance makes the study of the isochronicity much easier, since any isochronous potential G appears to be solution of a differential equation. More precisely, we state the following

Theorem B *Suppose hypothesis (\mathcal{H}) holds and let $g(x)$ be an analytic function and $G(x) = \int_0^x g(s)ds$. Then the equation*

$$\ddot{x} + g(x) = 0 \quad (1)$$

has an isochronous center at 0 if and only if

$$\frac{d}{dx}[G(x)/g^2(x)] = f(G) \quad (10)$$

where f is an analytic function defined in some neighborhood of 0.

We first deduce from Theorem B that the analytic functions f and g are naturally related and by (10) one has necessarily

$$\begin{aligned} f(0) &= -g''(0)/3, f'(0) = -(7/9)g'''(0) + g^{(4)}(0)/5, \\ f''(0) &= -\frac{1}{21}g^{(6)}(0) - \frac{310}{54}g''^{(5)}(0) + 2g''^{(2)}(0)g^{(4)}(0), \dots \end{aligned}$$

Proof of Theorem B Recall that

$$T(c) = \sqrt{2} \int_a^b \frac{dx}{\sqrt{c - G(x)}}$$

and by Lemma 2-1 its derivative $T'(c) = \frac{dT}{dc}$ may be written

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{g^2(x) - 2G(x)g'(x)}{g^2(x)\sqrt{c - G(x)}} dx \quad (11)$$

$$= \left(\frac{1}{c\sqrt{2}}\right) \int_a^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c - G(x)}} g(x) dx. \quad (12)$$

Suppose at first that $0 \in R^2$ is an isochronous center of system (2). Then, by Proposition 3-1

$$x - A(x) = 2\sqrt{2G(x)} \quad \text{for all} \quad 0 < x < b. \quad (13)$$

Deriving this expression

$$1 - A'(x) = \frac{g(x)\sqrt{2}}{\sqrt{G(x)}}$$

it implies

$$\frac{G(x)}{g^2(x)} = \frac{2}{(1 - A(x))^2}.$$

Therefore, by deriving with respect to x

$$\frac{d}{dx} \left[\frac{G(x)}{g^2(x)} \right] = \frac{d}{dx} \left[\frac{2}{(1 - A(x))^2} \right] = \frac{4A''(x)}{(1 - A'(x))^3} \quad (14)$$

On the other hand, we need the following

Lemma 3-1 *For any analytic involution $A(x)$ defined for all $x \in [a, b]$ the following expression holds*

$$\frac{A''(A(x))}{(1 - A'(A(x)))^3} = \frac{A''(x)}{(1 - A'(x))^3}$$

Proof Indeed, to prove this lemma we derive $A(A(x)) = x$ it yields

$$A'(A(x)) = \frac{1}{A'(x)} \quad \text{and} \quad A''(A(x)) = -\frac{A''(x)}{A'^3(x)}.$$

We now replace x by $A(x)$ in the right side of (9), one obtains

$$\frac{A''(A(x))}{(1 - A'(A(x)))^3} = -\frac{A''(x)}{A'^3(x)} \frac{1}{(1 - \frac{1}{A'(x)})^3}$$

So, since $A'(x) \neq 0$ one gets after simplification

$$\frac{4A''(A(x))}{(1 - A'(A(x)))^3} = \frac{4A''(x)}{(1 - A'(x))^3}$$

This lemma implies that expression (14) is A -invariant and therefore $\frac{d}{dx}[\frac{G(x)}{g^2(x)}]$ is an analytic function only dependent on G since $G(A(x)) = G(x)$.

We have now to prove the converse. We will use for that the following

Lemma 3-2 *The derivative of the period function $T(c)$ may be written as*

$$T'(c) = (\frac{1}{c\sqrt{2}}) \int_0^b \frac{\frac{d}{dx}(\frac{G}{g^2})(x) - \frac{d}{dx}(\frac{G}{g^2})(A(x))}{\sqrt{c - G(x)}} g(x) dx.$$

Proof Indeed, we have seen that

$$T'(c) = (\frac{1}{c\sqrt{2}}) \int_a^b \frac{\frac{d}{dx}(G/g^2)(x)}{\sqrt{c - G(x)}} g(x) dx$$

for $\bar{a} < a < 0$ and $0 < b < \bar{b}$.

By splitting the integral we get

$$\begin{aligned} T'(c) &= (\frac{1}{c\sqrt{2}}) \int_b^0 \frac{\frac{d}{dx}(G/g^2)(A(x))}{\sqrt{c - G(x)}} g(A(x)) A'(x) dx + (\frac{1}{c\sqrt{2}}) \int_0^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c - G(x)}} g(x) dx \\ &= (\frac{1}{c\sqrt{2}}) \int_b^0 \frac{\frac{d}{dx}(G/g^2)(A(x))}{\sqrt{c - G(x)}} g(x) dx + (\frac{1}{c\sqrt{2}}) \int_0^b \frac{\frac{d}{dx}(G/g^2)}{\sqrt{c - G(x)}} g(x) dx \\ &= (\frac{1}{c\sqrt{2}}) \int_0^b \frac{\frac{d}{dx}(\frac{G}{g^2})(x) - \frac{d}{dx}(\frac{G}{g^2})(A(x))}{\sqrt{c - G(x)}} g(x) dx. \end{aligned}$$

Thus, by this lemma condition (10) $\frac{d}{dx}[G(x)/g^2(x)] = f(G)$ implies necessarily that $\frac{d}{dx}[G(x)/g^2(x)] = \frac{d}{dx}[G(x)/g(A(x))^2]$ and then $T'(c) \equiv 0$. Theorem B is then proved.

Let us consider F the primitive of f such that $F(0) = 0$. Then

$$(10) \quad \frac{d}{dx}[G(x)/g^2(x)] = f(G(x)) \Leftrightarrow g(x) \frac{d}{dx}[G(x)/g^2(x)] = g(x)f(G(x)).$$

Integrate by parts, it yields $\frac{G(x)}{g(x)} - x = F(G(x))$. Therefore (10) is equivalent to $2G(x) - xg(x) = g(x)F(G(x))$. Thus, we get another criteria of isochronicity

Corollary 3-3 Under hypotheses of Theorem A, equation (1) admits an isochronous center at 0 if and only if

$$2G(x) - xg(x) = g(x)F(G(x)) \quad (10')$$

where F is an analytic function defined in some neighborhood of 0.

Some other equivalent conditions may also be deduced

Corollary 3-4 Under hypothesis (\mathcal{H}) , 0 is an isochronous center of (1) if and only if $x = x(G)$ is an analytic solution of the linear ODE

$$2G \frac{d^2x}{dG^2} + \frac{dx}{dG} = f(G), \quad (15)$$

where f is an analytic function.

Moreover, this solution must satisfy the conditions:

$$x(0) = 0, \quad \lim_{G \rightarrow 0} \left(\frac{x^2}{2G} \right) = 1.$$

Proof of Corollary 3-4 Let us consider again $F(t)$ the integral of $f(t)$. Then, it is easy to see that condition $\frac{d}{dx}(G/g^2) = f(G)$ is equivalent to

$$2G \frac{dx}{dG} = x + F(G). \quad (16)$$

We derive (16) with respect to the variable G , we then obtain (15).

So, for any analytic function f the linear equation (15) admits a unique solution $x = x(G)$ according to initial conditions. More precisely, consider the change $x = \sqrt{2G} + y$. Then $y = y(G)$ is solution of

$$2G \frac{d^2y}{dG^2} + \frac{dy}{dG} = f(G)$$

with initial points : $y(0) = 0, y'(0) = 1$. A resolution of the last equation yields

$$y(G) = \sqrt{2G} \int_0^G \frac{F(\nu)}{(2\nu)^{3/2}} d\nu.$$

Thus, a solution of (15) may be written

$$x(G) = \sqrt{2G} \left(1 + \int_0^G \frac{F(\nu)}{(2\nu)^{3/2}} d\nu \right).$$

Let us consider now the involution A such that $G(A(x)) = G(x)$. Another consequence of Theorem B is

Corollary 3-5 *Let $G(x) = \int_0^x g(s)ds$ be an analytic potential of the scalar equation*

$$\ddot{x} + g(x) = 0. \quad (1)$$

Let $A(x)$ be an analytic involution defined by:

$$G(A(x)) = G(x) \text{ and } A(x)x < 0.$$

Then 0 is an isochronous center of (1) if and only if

$$\frac{d}{dx}[G(x)/g^2(x)] = \frac{4A''(x)}{(1 - A'(x))^3}.$$

Moreover, the last expression is A -invariant.

Proof of Corollary 3-5 Let A be an analytic involution then $G(A(x)) = G(x)$ implies

$$\frac{dG}{dx} = g(x) = \frac{dA}{dx}g(A(x))$$

and

$$\frac{G}{g^2}(A(x)) = \left(\frac{dA}{dx}\right)^2 \frac{G}{g^2}(x).$$

Deriving the last expression we then obtain

$$A'(x) \frac{d(\frac{G}{g^2})}{dx}(A(x)) = 2A'(x)A''(x) \frac{G}{g^2}(x) + A'^2(x) \frac{d(\frac{G}{g^2})}{dx}(x).$$

Since $A'(x) \neq 0$ then, after simplification we get the differential equation

$$\frac{d(\frac{G}{g^2})}{dx}(A(x)) = 2A''(x) \frac{G}{g^2}(x) + A'(x) \frac{d(\frac{G}{g^2})}{dx}(x). \quad (17)$$

By Theorem B,

$$\frac{d}{dx}[G(x)/g^2(x)] = f(G) \text{ implies } \frac{d}{dx}[G(x)/g^2(x)] = \frac{d}{dx}[G(x)/g(A(x))^2].$$

Therefore, the solution of equation (17) is

$$\frac{G}{g^2}(x) = \frac{2}{(1 - A'(x))^2} \quad (18)$$

since $A'(0) = -1$. Thus, by deriving one gets $\frac{d}{dx}[G(x)/g^2(x)] = \frac{4A''(x)}{(1-A'(x))^3}$.

To prove the converse we require again Lemma 3-2

$$T'(c) = \left(\frac{1}{c\sqrt{2}}\right) \int_0^b \frac{\frac{d}{dx}\left(\frac{G}{g^2}(x)\right) - \frac{d}{dx}\left(\frac{G}{g^2}(A(x))\right)}{\sqrt{c - G(x)}} g(x) dx.$$

The analytic involution $A(x)$ can also be defined as a solution of a linear ODE. The following result is analogous to Corollary 3-4.

Corollary 3-6 *Under hypotheses of Theorem A, equation (1) admits an isochronous center at 0 if and only if the involution $A = A(G)$ is a solution of*

$$2G \frac{dA}{dG} = A(G) + F(G) \quad (19)$$

f is an analytic function and F is its integral.

Moreover, this solution must satisfy the conditions:

$$A(0) = 0, \quad \lim_{G \rightarrow 0} \left(\frac{A^2}{2G}\right) = 1$$

Proof of Corollary 3-6 To see that, we start from $A(A(x)) = x$ and $G(A(x)) = G(x)$. So equation (11) $2G \frac{dx}{dG} = x + F(G)$ is equivalent to

$$2G(A(x)) = \frac{dG(A(x))}{dx} [A(x) + F(G(A(x)))] = \frac{dG(x)}{dx} (x + F(G)).$$

On the other hand, deriving $G(A(x)) = G(x)$ with respect to x we get $\left(\frac{dA(x)}{dx}\right)\left(\frac{dG(A(x))}{dx}\right) = \frac{dG(x)}{dx}$. We then obtain

$$A(x) + F(G(x)) = [x + F(G(x))] \frac{dA(x)}{dx}.$$

Hence we deduce the ODE (19).

Remark 3-7 The method of Urabe requires the use an intermediary function h that is not in general explicitly known. Indeed, in order to determine the potential isochrones one must be able to show by means of the change of variables $X = \sqrt{2G}$ that the inverse function of $G(x) = y$ is of the form $x = X + H(X)$. What is not easy to achieve in any case. On the other hand, our approach is more direct. Our criteria are simply relations between an isochronous potential and its derivative. To be clearer, condition (10) of Theorem B is a sort of non linear differential equation of order one with respect to the variable x . All solution $G = G(x)$ of (10) $\frac{d}{dx}[G(x)/g^2(x)] = f(G(x))$ or equivalently (10') $2G(x) - xg(x) = g(x)F(G(x))$ provides an isochronous potential verifying $G(0) = G'(0) = 0, G''(0) = 1$.

It is well known that the harmonic potential $G = \frac{1}{2}x^2$ is the only polynomial potential which is isochronous. Concerning the rational case, Chalykh and Veselov [2] proved the following

Proposition 3-8 ([2]) *Under hypothesis (\mathcal{H}) a rational potential $G(x)$ (which is not a polynomial) is isochronous if and only if*

$$G(x) = \frac{1}{8\alpha^2}[\alpha x + 1 - \frac{1}{\alpha x + 1}]^2.$$

The proof we give below is different of that given by [2].

Proof of Proposition 3-8 Let us recall that for a involution A we get $G(A(x)) = G(x)$ and $A(x)x < 0$ for all $x \in [a, b]$. By Proposition 3-1 this potential is isochronous if

$$x - A(x) = 2\sqrt{2G(x)} \quad \text{for all} \quad 0 < x < b$$

or equivalently it verifies the functional equation

$$G(x) = G(x - 2\sqrt{2G(x)}) \quad \text{for all} \quad 0 < x < b.$$

When the potential is rational it is required that $G(x)$ has to be the square of a rational function of the form

$$G(x) = \frac{1}{2}(\frac{xP(x)}{Q(x)})^2$$

where $P(x)$ and $Q(x)$ are polynomials without common zeros.

This means the involution has to be meromorphic and has the same poles as $G(x)$. It can be written

$$A(x) = x - \left(\frac{2xP(x)}{Q(x)} \right) = \frac{xQ(x) - 2xP(x)}{Q(x)}$$

But, it is known (see for example Theorem (15.4) p. 296 of [6]) that the only meromorphic functions verifying $A^2(x) = x$ are $A(x) = L^{-1}(\frac{a}{L(x)})$ where $L(x) = ax + b$ a affine function since other meromorphic functions are not invertible. A is then an homographic function

Thus, by hypothesis (\mathcal{H}) and since $A(0) = 0$ and $A'(0) = -1$ we get

$$A(x) = -\frac{x}{\alpha x + 1}$$

implying

$$\frac{2P(x)}{Q(x)} = 1 + \frac{1}{\alpha x + 1} = \frac{\alpha x + 2}{\alpha x + 1}.$$

4 Parametrization of isochronous centers

In this part we suppose that all functions are analytic. In particular, let us write

$$g(x) = x + \sum_{n \geq 2} a_n x^n \quad \text{and} \quad G(x) = \frac{1}{2}x^2 + \sum_{n \geq 2} \frac{a_n}{n+1} x^{n+1},$$

and suppose that r_0 is the radius of convergence of these power series. By the Cauchy-Hadamard formula $\frac{1}{r_0} = \lim_{n \rightarrow \infty} |a_n|^{1/n}$.

The purpose of this section is to highlight conditions so that an analytic potential $G(x)$ be isochronous. To that end let us write

$$G(x) = \frac{1}{2}x^2 + G_1(x) + G_2(x)$$

where the function $G_1(x) = \sum_{k \geq 2} \frac{a_{2k-2}}{2k-1} x^{2k-1}$ is odd and $G_2(x) = \sum_{k \geq 2} \frac{a_{2k-1}}{2k} x^{2k}$ is even.

We looking for general conditions on coefficients a_n ensuring the isochronicity of the center 0 of equation (1) $\ddot{x} + g(x) = 0$. Recall that only the harmonic potential $G(x) = \frac{1}{2}x^2$ is an even isochronous potential ($G_1(x) \equiv 0$). That means the odd coefficients a_{2k+1} cannot be all zero when the isochronous potential is non-harmonic. In fact, we will prove little more.

Starting from Theorem A we will show there are infinitely many necessary conditions verifying by the coefficients in order the potential $G(x)$ be isochronous. More precisely, we state that the even coefficients a_{2k} may be free and the odd coefficients a_{2k+1} are polynomials with respect to a_{2k} .

Theorem C *Let the analytic potential*

$$G(x) = \frac{1}{2}x^2 + \sum_{n \geq 3} \frac{a_{n-1}}{n} x^n = \frac{1}{2}x^2 + G_1(x) + G_2(x)$$

of equation

$$(1) \quad \ddot{x} + g(x) = 0.$$

When the equation (1) has an isochronous center at 0 then the odd coefficients of the expansion of $g(x)$ can be expressed in terms of rational polynomials involving the even coefficients:

$$a_{2k+1} = f(a_{2k}, a_{2k-2}, \dots, a_2).$$

In particular, when the potential $G(x)$ is isochronous then $G_2(x) \equiv 0$ is equivalent to $G_1(x) \equiv 0$, i.e. $G(x) = \frac{1}{2}x^2$ is harmonic.

Thanks to *Maple* we are able to calculate the first terms :

$$\begin{aligned} a_3 &= \frac{10}{9}a_2^2, & a_5 &= \frac{14}{5}a_2a_4 - \frac{56}{27}a_2^4, \\ a_7 &= \frac{-592}{45}a_4a_2^3 + \frac{848}{81}a_2^6 + \frac{24}{7}a_2a_6 + \frac{36}{25}a_4^2, \\ a_9 &= \frac{110}{27}a_2a_8 - \frac{440}{21}a_2^3a_6 + \frac{27808}{243}a_2^5a_4 - \frac{536800}{6561}a_2^8 - \frac{1144}{45}a_2^2a_4^2 + \frac{22}{7}a_4a_6 \end{aligned}$$

$$a_{11} = \frac{52}{11} a_2 a_{10} + \frac{57616}{135} a_2^4 a_4^2 - \frac{2600}{81} a_2^3 a_8 + \frac{125008}{567} a_2^5 a_6 - \frac{4837664}{3645} a_2^7 a_4 + \frac{5631808}{6561} a_2^{10} - \frac{2392}{125} a_2 a_4^3 - \frac{7384}{105} a_2^2 a_4 a_6 + \frac{52}{15} a_4 a_8 + \frac{78}{49} a_6^2$$

$$a_{13} = -72 a_2 a_6 a_4^2 - \frac{2632}{27} a_2^2 a_4 a_8 + \frac{38176}{27} a_2^4 a_4 a_6 + \frac{70}{13} a_2 a_{12} + \frac{42}{11} a_4 a_{10} + 10/3 a_6 a_8 - \frac{9430624}{1215} a_4^2 a_2^6 + \frac{375769408}{19683} a_4 a_2^9 + \frac{166544}{225} a_4^3 a_2^3 - \frac{920}{21} a_2^2 a_6^2 - \frac{2190080}{729} a_2^7 a_6 - \frac{14000}{297} a_2^3 a_{10} + \frac{300944}{729} a_2^5 a_8 - \frac{74681600}{6561} a_2^{12} - \frac{616}{125} a_4^4$$

Proof of Theorem C Let the analytic function

$$g(x) = x + \sum_{k=2}^n a_k x^k.$$

By Theorem A the potential $G(x) = \frac{1}{2}x^2 + \sum_{n \geq 3} \frac{a_{n-1}}{n} x^n$ is isochronous if and only if the following equality holds (10) $\frac{d}{dx}(\frac{G(x)}{g^2(x)}) = f(G)$ where f is an analytic function, set

$$f(G) = b_0 + b_1 G + b_2 G^2 + b_3 G^3 + \dots$$

After replacing and equaling the two sides of (10) :

$$\left(x + \sum_{k=2}^n a_k x^k\right)^{-1} - \left(x^2 + \sum_{k=2}^n \frac{2a_k}{k+1} x^{k+1}\right) \left(1 + \sum_{k=2}^n \frac{a_k x^k}{x}\right) \left(x + \sum_{k=2}^n a_k x^k\right)^{-3} = \sum_{k=0}^p b_k \left(\frac{1}{2} x^2 + \sum_{k=2}^n \frac{a_k}{k+1} x^{k+1}\right)^k.$$

Then we identify the analytic expansions of the two expressions. The unknown coefficients will then be determined by comparing powers in x :

$$b_0 = -\frac{2}{3}a_2, \quad a_3 = \frac{10}{9}a_2^2 \quad b_1 = 11a_2a_3 - 6a_2^2 - \frac{24}{5}a_4$$

$$-(10/3)a_5 - (1/3)b_1a_2 + \frac{116}{15}a_2a_4 + (14/3)a_2^4 + 4a_3^2 - 13a_3a_2^2 = 0, \\ -\frac{30}{7}a_6 - \frac{1}{4}b_2 - \frac{1}{4}b_1a_3 + 10a_2a_5 - \frac{20}{3}a_2^5 + \frac{21}{2}a_4a_3 - 17a_4a_2^2 - \frac{35}{2}a_2a_3^2 + 25a_3a_2^3 = 0, \dots$$

From these recursion formulae the coefficients can be easily determined. After eliminating b_0, b_1, b_2, \dots we then deduce the expressions of coefficients

a_3, a_5, a_7, \dots

The last part of Theorem C will be proved by recurrence. Suppose that all coefficients $a_n = 0$ for any $n < 2p$ where p is a positive integer and $b_0 = b_1 = b_2 = \dots = b_{p-1} = 0$. So, we may write

$$g(x) = x + a_{2p}x^{2p} + a_{2p+1}x^{2p+1} + a_{2p+2}x^{2p+2} + \dots$$

and

$$G(x) = \frac{1}{2}x^2 + \frac{a_{2p}}{2p+1}x^{2p+1} + \frac{a_{2p+1}}{2p+2}x^{2p+1} + \frac{a_{2p+2}}{2p+1}x^{2p+3} + \dots$$

Calculate

$$\frac{d}{dx}\left(\frac{G}{g^2}\right) = -2p a_{2p} x^{2p-1} - a_{2p+1} x^{2p} (2p+1) + a_{2p} x^{2p+1} + a_{2p+1} x^{2p+2} + \dots$$

and equating with

$$f(G) = b_p G^p + b_{p+1} G^{p+1} + \dots = \frac{b_p}{2^p} x^{2p} + \frac{b_{p+1}}{2^{p+1}} x^{2p+2} + \frac{b_{p+2}}{2^{p+2}} x^{2p+4} + \dots$$

Thus, necessarily $a_{2p} = 0$.

Remarks 4-1

Consider again Theorem B then equality

$$(10) \quad \frac{d}{dx}(G/g^2) = b_0 + b_1 G + b_2 G^2 + b_3 G^3 + b_4 G^4 + \dots$$

ensures the isochronicity of the potential G .

It is also possible to express coefficients a_n of the analytic expansion of $g(x)$ (or its successive derivatives at 0) in terms of b_0, b_1, b_3, \dots

Indeed,

$$\begin{aligned} g'(0) &= 1, g''(0) = -3b_0, g'''(0) = 15b_0^2, g^{(4)}(0) = -105b_0^3 - 30b_1, \\ g^{(5)}(0) &= 945b_0^4 + 630b_0b_1, g^{(6)}(0) = -10395b_0^5 - 11340b_0^2b_1 - 840b_2, \\ g^{(7)}(0) &= 207900b_0^3b_1 + 30240b_0b_2 + 135135b_0^6 + 11340b_1^3, \dots \end{aligned}$$

g may also be written

$$\begin{aligned} g(x) &= x - \frac{3}{2} b_0 x^2 + \frac{5}{2} b_0^2 x^3 + \left(-\frac{5}{24} b_1 - \frac{35}{8} b_0^3\right) x^4 + \frac{7}{45} b_0 \left(\frac{405}{8} b_0^3 + \frac{45}{8} b_1\right) x^5 + \\ &\left(-\frac{7}{120} b_2 - \frac{21}{8} b_0^2 b_1 - \frac{231}{16} b_0^5\right) x^6 + \left(\frac{55}{8} b_0^3 b_1 + \frac{429}{16} b_0^6 + \frac{3}{10} b_0 b_2 + \frac{1}{16} b_1^2\right) x^7 + \dots \end{aligned}$$

and the equation $\ddot{x} + g(x) = 0$ has an isochronous center at 0 for parameters values b_0, b_1, b_2, \dots

Thus, any isochronous potential G is a multiparameters analytic function which may be expressed under the general form

$$G(x) = \frac{1}{2}x^2 - \frac{1}{2}b_0x^3 + \frac{5}{8}b_0^2x^4 + \left(-\frac{1}{24}b_1 - \frac{7}{8}b_0^3\right)x^5 + \frac{7}{270}b_0\left(\frac{405}{8}b_0^3 + \frac{45}{8}b_1\right)x^6 + \frac{1}{7}\left(-\frac{7}{120}b_2 - \frac{21}{8}b_0^2b_1 - \frac{231}{16}b_0^5\right)x^7 + \frac{1}{8}\left(\frac{55}{8}b_0^3b_1 + \frac{429}{16}b_0^6 + \frac{3}{10}b_0b_2 + \frac{1}{16}b_1^2\right)x^8 + \dots$$

A general expression of a_{2p} as a polynomial of b_0, b_1, b_2, \dots seems difficult to built. Nevertheless, it is possible to know the first coefficient

$$a_{2p+1} = \frac{-1}{2^{p-1}} \frac{2p+1}{(2p)(2p-1)} b_{p-1} + \dots$$

5 Application to the search of isochronous potentials

In this part, we will see that above results allow us to determinate families of isochronous potentials. More precisely, we then apply Theorems B to produce potentials with constant period. Some of them are new

A three-parameters family of isochronous potentials

To be concrete, we derive at first a three-parameters family of potentials which appears little more general than the one given by Dorignac [6].

Let us consider the following case

$$\frac{d}{dx}\left(\frac{G}{g^2}\right) = \frac{\alpha}{(1 + \beta G)^{3/2}},$$

where α and β are real parameters such that $2\alpha^2 \leq \beta$.

Thanks to *Maple* a resolution of these equations yields

$$G(x) = \frac{8\alpha^2 + (\beta + 2\alpha^2)(4\alpha x + \beta x^2) - (4\alpha^2 + 2\alpha\beta x)\sqrt{2(2 + \beta x^2 + 4\alpha x)}}{2(\beta - 2\alpha^2)^2} \quad (20)$$

Then, the above potential is isochronous according to Theorem B. It may also be written

$$G(x) = \frac{1}{2}X^2$$

where

$$X = \frac{2\alpha + \beta x - \alpha\sqrt{2(2 + \beta x^2 + 4\alpha x)}}{\beta - 2\alpha^2}.$$

The function h of Proposition 3-4 is

$$h(X) = \frac{\alpha\sqrt{2}X}{\sqrt{2 + \beta X^2}}.$$

Recall that X is such that $\frac{dX}{dx} > 0$ so that X is a bijection of x . Moreover, $h(X)$ is odd function verifying $|h(X)| < 1$ for any $X \in R$ since $2\alpha^2 \leq \beta$.

Applying scaling property of isochronous potentials, (Claim 2, Corollary of [6]). The potentials $G(x)$ and $\frac{1}{\gamma^2}G(\gamma x)$ have the same period. That means the following three-parameters potentials family is isochronous

$$G(x) = \frac{1}{2\gamma^2}X^2(\gamma x) = \frac{[2\alpha + \beta\gamma x - \alpha\sqrt{2(2 + \beta\gamma^2 x^2 + 4\alpha\gamma x)}]^2}{2\gamma^2(\beta - 2\alpha^2)^2} \quad (21)$$

The derivative of $G(x)$ is then

$$g(x) = \frac{2\alpha + \beta\gamma x - \alpha[\sqrt{2(2 + \beta\gamma^2 x^2 + 4\alpha\gamma x)}](\beta\gamma - \frac{\alpha(4\beta\gamma^2 x + 8\alpha\gamma)}{\sqrt{2(2 + \beta\gamma^2 x^2 + 4\alpha\gamma x)}})}{\gamma^2(\beta - 2\alpha^2)^2}$$

The involution A defined by $G(A(x)) = G(x)$, $A(x)x < 0$ may be written

$$A(x) = x - 2\frac{[2\alpha + \beta\gamma x - \alpha\sqrt{2(2 + \beta\gamma^2 x^2 + 4\alpha\gamma x)}]}{\gamma(\beta - 2\alpha^2)}$$

As special cases we may derive the following

1 - For any $\gamma \neq 0$ and $\beta = 2\alpha$ we obtains the two-parameters family of isochronous potentials (see (24) of [6]) introduced for the first time by Stillinger and Stillinger [12]. Indeed, after replacing in (21) β by 2α and simplifying by 2α it yields

$$G(x) = \frac{[1 + \gamma x - \sqrt{1 + \alpha\gamma^2 x^2 + 2\alpha\gamma x}]^2}{2(1 - \alpha)^2} \quad (22)$$

2 - The case $\alpha = \beta = 0$ and $\gamma = 1$ yields the harmonic potential : $G(x) = \frac{1}{2}x^2$.

3 - The case $\beta = 0$ and $\gamma = 1$ gives the Urabe potential (see Corollary 2-2) :

$$G(x) = \frac{4}{\alpha^2} - \frac{2}{\alpha}(x + 2\frac{\sqrt{1 - \alpha x}}{\alpha}).$$

4 - The case $2\alpha = \beta$ and $\gamma = 1$ yields the Bolotin-Mc Kay potential (see [6]). Indeed,

$$\frac{d}{dx}(\frac{G}{g^2}) = \frac{\alpha}{(1 + 2\alpha G)^{3/2}}$$

implies

$$G(x) = \frac{[1 + x - \sqrt{1 + \alpha x^2 + 2\alpha x}]^2}{2(1 - \alpha)^2}$$

We then deduce the function h of Proposition 3-4

$$h(X) = \frac{\alpha X}{\sqrt{1 + X^2 \alpha}}$$

5 - The case $2\alpha^2 = \beta$ and $\gamma = 1$ yields the isotonic potential (see [6]) Thanks to *Maple* a resolution of

$$\frac{d}{dx}(\frac{G}{g^2}) = \frac{\alpha}{(1 + \alpha^2 G)^{3/2}}$$

gives

$$G = \frac{1}{4} \frac{x^2(2 + \alpha x)^2}{(1 + \alpha x)^2}.$$

The function h of Proposition 3-4 is

$$h(X) = \frac{\alpha X}{\sqrt{1 + X^2 \alpha^2}}$$

and its integral is

$$H(X) = \frac{\sqrt{1 + X^2 \alpha^2}}{\alpha} - \frac{1}{\alpha} = x - X$$

Thus

$$\frac{1 + X^2 \alpha^2}{\alpha^2} = \left(x - X + \alpha^{-1}\right)^2$$

We then obtain by another way

$$G = \frac{1}{4} \frac{x^2(2 + \alpha x)^2}{(1 + \alpha x)^2} = \frac{1}{8\alpha^2} \left[\alpha x + 1 - \frac{1}{\alpha x + 1} \right] 2$$

Others isochronous potentials

We will give others two-parameters families of potentials with constant period which seems to be new in the literature.

1 - Let us consider

$$\frac{d}{dx} \left(\frac{G}{g^2} \right) = \frac{\alpha}{(1 + 2\beta^2 G)^{5/2}}.$$

Then the function h of Proposition 3-4 may be calculated

$$h(X) = \alpha \left(1/3 \frac{X}{(1 + X^2 \beta^2)^{3/2}} + 2/3 \frac{X}{\sqrt{1 + X^2 \beta^2}} \right)$$

and its integral is

$$H(X) = 1/3 \frac{\alpha (1 + 2 X^2 \beta^2)}{\beta^2 \sqrt{1 + X^2 \beta^2}} - \frac{\alpha}{3\beta^2}$$

Taking for example $\beta^2 = 2\alpha/3$ and thanks to *Maple* we find another isochronous potential.

Indeed, since $x = X + H(X)$ and solving $H(X)^2 = (x - X)^2$ we get

$$X(x) = (1/24) \frac{[u(x)]^{1/3}}{\alpha(3+4x\alpha)} + (1/24) \frac{(16x^2\alpha^2+24x\alpha-423)(3+4x\alpha)}{\alpha[u(x)]^{1/3}} + (1/24) \frac{3+4x\alpha}{\alpha}$$

where

$$u(x) = -5751 + 31536x\alpha + 21600x^2\alpha^2 + 768x^3\alpha^3 + 256x^4\alpha^4 + 72\sqrt{3}\sqrt{1024x^6\alpha^6 + 4608x^5\alpha^5 + 26496x^4\alpha^4 + 62208x^3\alpha^3 + 101412x^2\alpha^2 + 86022x\alpha + 45927}$$

So, according to Theorem A and in using scaling properties the potential

$$G(x) = \left(\frac{1}{2}\right)\left(\frac{1}{24}\right)^2 \left[\frac{[u(\gamma x)]^{1/3}}{\alpha(3+4\gamma x\alpha)} + \frac{(16\gamma^2x^2\alpha^2+24\gamma x\alpha-423)(3+4\gamma x\alpha)}{\alpha[u(\gamma x)]^{1/3}} + \frac{3+4\gamma x\alpha}{\alpha} \right]^2$$

is isochronous.

2 - Let

$$\frac{d}{dx} \left(\frac{G}{g^2} \right) = \frac{2\alpha + 2\alpha^3 G}{(1 + 2\alpha^2 G)^{5/2}}.$$

The function h is then

$$h(X) = \frac{2\alpha X + \alpha^3 X^3}{(1 + \alpha^2 X^2)}$$

and its integral has the following simple form

$$H(X) = \frac{\alpha X^2}{\sqrt{1 + \alpha^2 X^2}}.$$

Thanks to *Maple* we find

$$X = \frac{1}{6\alpha^2 x} v(x) + \frac{(-10\alpha^2 x^2 + 1 + \alpha^4 x^4)}{6\alpha^2 x v(x)} + \frac{1 + \alpha^2 x^2}{6\alpha^2 x}$$

where

$$v(x) = [-15\alpha^2 x^2 + 39\alpha^4 x^4 + 1 + \alpha^6 x^6 + \alpha^3 x^3 \sqrt{(-3 + 33\alpha^2 x^2 + 3\alpha^4 x^4)}]^{1/3}$$

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